USING THE INFINITE DESCENT METHOD TO FIND CONVENIENT RATIONAL AND NON-RATIONAL NUMBERS USING DEDEKIND CUTS

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Abstract

It is well-known that irrational numbers play a relevant role in mathematics and basic sciences e.g., the number introduced by the Babylonians and Egyptians of ancient times, Euler´s number e to explain exponentially-varying processes, and the λ of Conway´s cosmological theory. Therefore, a strong understanding of real numbers is important. Many mathematicians such as R. Dedekind and W. Rudin, when introducing the real numbers via the rational and irrational numbers and the concept of Dedekind cuts, make use of "convenient numbers" such as Rudin's h which seem to be "taken out of a hat." From a pedagogical point of view, the use of these numbers has proven to be a sticky issue to both students and professors because there has been little, if any, justification for their "convenience". In this paper the authors, using Dedekind cuts explain the introduction of those "convenient" numbers using the infinite descent method. The Extended Euclidean Convergent Algorithm is used to create convergent fractions to approximate irrational numbers with a desired approximation via the computer.

Keywords: Dedekind cuts, Infinite descent, Well-ordering Principle, Extended Euclidean Convergent Algorithm, Irrationals

JEL Classification: C

1. Introduction

In elementary arithmetic we learn about the decimal expression of rational numbers: a rational number has either a finite decimal expression or an infinite periodic one. It is well known that rational numbers, its set denoted by Q, are inadequate to solve certain algebraic problems such as the existence of rational numbers which square equals 2 [1], or the existence of a rational solution of the quintic equation $x^5 - x - 1 = 0$ (whose only real root is 1.1673039782614186843…). Thus, the decimal expression of an irrational is neither finite nor ever becomes periodic. In his quest to introduce the real numbers using algebraic methods, Dedekind introduced the notion of *cuts* [2]. As stated by Dedekind, every rational number a divides Q into two classes A_1 and A_2. The class A_1 is the set of rational numbers a_1 less than every number a_2 of the class A_2. The rational number a itself can be associated with either class so it could be the greatest number of A_1 or the least of A_2. Dedekind called any separation of the rational numbers into two classes, as just described, a cut or *schnitt* in its original German terminology. Dedekind

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also pointed out that there are cuts not produced by rational numbers such as the ones produced by integers which are not a perfect square of any other integer e.g., the integer *D=2* as shown later in this work.

Irrational numbers are important not because of the mere impossibility of expressing them exactly as ratios of integers, but also because they play important roles in mathematics, geometry, and natural sciences. Such is the case of the number π = 3.1415926535...for which tables with 1000 or more decimals have been published and which the ancients already considered worth to calculate. Notably, archeologists have discovered also that ancient Babylonians and Egyptians succeeded writing π with 2-3 decimals. Many other examples of relevant irrationals can be presented, among them Euler´s number *e*, the base of natural logarithms and key to explain all exponentially varying processes, and Conway´s constant λ=1.303577269034296 [3]. In passing, these three irrationals are said to be transcendentals also. That is, they are not roots of any non-zero polynomial which coefficients are rational numbers. In Section 2 we present the Dedekind non-rational cuts. Section 3 is devoted to Rudin´s treatment of cuts showing the existence of "gaps" in *Q*. Section 4 considers the basic representation of integers in two's complement notation and indicates some of the problems of representing digitally both rationales and irrational numbers in a computer. In Section 5 we consider the rational approximations to irrational numbers. Section 6 is devoted to the Euclidean Convergent Algorithm and its application to obtain rational approximations to any irrational number using a computer. Finally, Section 7 is devoted to our concluding remarks.

2. The Existence of Non-rational cuts

The following theorem illustrates the existence of infinitely many cuts not produced by rational numbers as indicated by Dedekind. The authors have amplified the original proof and have clarified some of its details. From now on we will use the standard notation for denoting the set of integers and positive integers as *Z* and Z^+ respectively. Likewise, we denote the positive rational numbers as Q^+.

Theorem 1 "Every positive integer *D* which is not a perfect square of any other integer lies between the square of two consecutive positive integers, moreover *D* is not a perfect square of any rational number."

As stated, the thesis of this theorem consists of two parts. The first is concerned with a relationship between integer numbers (*D* and two other integers). The second concerns the nature of the number *D* itself which is not a perfect square of any rational number. We will address these two issues in order.

Proof of part 1 (by the method of contradiction or reduction ad absurdum)

"Every positive integer *D* which is not a perfect square of any other integer lies between the square of two consecutive positive integers."

This statement can be translated into standard mathematical notation as follows:

$$(\forall D \in Z^+)\ (\forall a \in Z)\ D \neq a^2 \rightarrow (\exists \lambda \in Z^+)\ \lambda^2 < D < (\lambda + 1)^2$$

To prove it let's consider the set of all positive integers which square is less than *D*. This set is clearly non-empty and has a maximum element λ. Therefore, we can write $\lambda^2 < D$. As required by the method of contradiction let's begin by writing the negation of the thesis. Therefore, using De Morgan's laws [4] we get:

$$(\forall \lambda \in Z^+)\ \lambda^2 \geq D \quad or \quad D \geq (\lambda + 1)^2$$

By the mere definition of the integer λ given above, the first inequality $\lambda^2 \geq D$ cannot be satisfied. Now, let's consider the second inequality $D \geq (\lambda + 1)^2$. This last statement cannot be satisfied either because $(\lambda + 1) > \lambda$ and we have assumed that λ is the maximum integer which square is strictly less than *D*. Therefore, *D* lies between the two integers λ^2 and $(\lambda + 1)^2$. Q.E.D.

Thus, this positive integer *D* divides the set *Q* into two *classes*: *Class* A_1 contains all positive rational numbers a_1 such that their squares $a_1^2 < D$, and *Class* A_2 containing all other rational numbers. Interestingly, this number *D* itself cannot be the square of any rational number as we now demonstrate in the second part of the theorem.

Proof of part 2 (by the method of infinite descent)

"There is no rational number which square equals *D*"

A proof by the method of infinite descent is also a type of proof by contradiction. The basic difference between a proof by this method and a standard proof by contradiction is that, in an infinite descent proof, we look for a sequence of infinite decreasing positive numbers that satisfies a previously defined condition. Because the positive integer numbers have a least element according to the Well-ordering Principle [5] an infinite sequence of decreasing positive numbers is not possible. The proof we present below follows Dedekind's [2] but has been expanded to clarify some of its steps and conclusion.

As indicated above we begin by negating the thesis. That is, we assume that there is a rational number p/q with $q \neq 0$ such that $(p/q)^2 = p^2/q^2 = D$ where, without loss of generality, we can assume that *p* and *q* are both positive integers and their gcd(*p,q*) = 1. The latter expression for *D* can be rewritten in quadratic-equation form as

$$p^2 - Dq^2 = 0$$

According to the infinite descent method, what we need to find is a new rational number numerically equivalent to *D* which denominator is less than *q*.

Using a positive integer λ it is possible to obtain a relationship between p and q of the following form:

$$\lambda q < p < (\lambda + 1)q$$

In fact, this latter relationship can be obtained as shown next. From part 1 of the theorem, we already showed that

$$\lambda^2 < D < (\lambda + 1)^2,$$

replacing D by its equivalent $(p/q)^2$ in this last inequality we may obtain

$$\lambda^2 < (p/q)^2 < (\lambda + 1)^2 \Leftrightarrow \lambda^2 q^2 < p^2 < q^2 (\lambda + 1)^2 \Leftrightarrow \lambda q < p < (\lambda + 1)q$$

Subtracting λq from each member of this last inequality we get

$$0 < p - \lambda q < q$$

This inequality shows that $p - \lambda q$ is a positive integer less than q. Let's call q' this positive integer. That is, $q' = p - \lambda q$. We will use q' as the denominator of the new rational number being sought. Because the denominator of this new rational number is less than q (the initial denominator of D) the numerator p' of the new rational number must be also less than the numerator p of D. This is so because the numbers must be numerically equivalent. This new numerator should be positive and of the following form:

$$p' = \frac{p(p - \lambda q)}{q}$$

That p' is less than p, the numerator assumed for D, can be easily demonstrated. In fact, knowing already that $0 < p - \lambda q < q$ and dividing this inequality by positive q, we obtain

$$0 < \frac{p - \lambda q}{q} < 1$$

Multiplying the last inequality by positive p and considering the actual value of p' defined a few lines above we have that

$$p' = \frac{p\,(p - \lambda q)}{q} < p$$

This last inequality indicates that p' is less than p as we wanted to show.

'The values of p' and q' just found satisfy the quadratic-equation stated before for D. That is, $p'^2 - D\, q'^2 = 0$. Therefore, a new rational number numerically to equivalent to D and with a smaller denominator q' ($q' < q$) has been found. Continuous repetition of this procedure will allow us to find an infinite sequence of decreasing integers q', q'', q'''… which, as we indicated before is not possible due to the Well-ordering Principle. From this contradiction, we can conclude that our hypothesis about D being the square of a rational is false. Therefore, D is an irrational number. Q.E.D.

3. Rudin rational cuts

We now want to use the concept of *Dedekind cuts* to show explicitly that the set Q of rational numbers does contain "gaps" such as the lack of a rational number whose square is a positive integer D. Consider then the rather familiar case $D=2$ and state our next theorem as follows:

Theorem 2 "Let A be the set of all positive rational numbers p such that $p^2 < 2$, and B the set of all positive rational numbers p such that $p^2 > 2$ where A and B are rational Dedekind cuts in Q."

Like Theorem 1, this new theorem consists of two parts, one for set A and one for set B. We will address them in that order.

Proof of part 1 (for Set A)

Proof: Consider first a positive rational $p \in A \subset Q$, therefore, $p^2<2$. Take now another rational q such that $q>p$, and let $q = p + h$, (Figure 1), h being a rational such that $0 < h < 1$. The argument is to add this h to p is to get a larger q that would still belong to A. We then have:

$$q^2 = p^2 + 2ph + h^2 = p^2 + (2p + h)h,$$

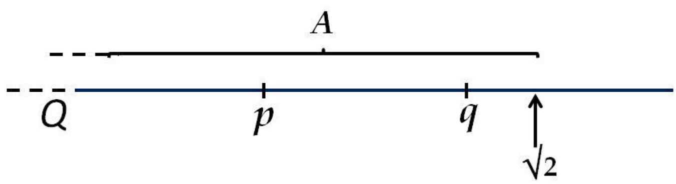

Figure 1

p lies in the set $A \subset Q$ of rational numbers which square p^2 is less than 2; q is a rational such that $p<q$ and $q - p = h$.

Since $0<h<1$ we may rewrite the last equation as:

$$q^2 < p^2 + (2p+1)h$$

Because we want to show that $q \in A$ we need to eliminate the factor $(2p+1)$ as well as p^2 from the last inequality. A little thought led us to write the number h as fraction of the form $(2-p^2)/(2p+1)$. Replacing this value in the last inequality we obtain

$$q^2 < p^2 + (2p+1)\left[\frac{2-p^2}{2p+1}\right] = p^2 + (2-p^2) = 2 \quad \Leftrightarrow q^2 < 2$$

Therefore, the rational $q>p$ that was introduced above does belong in A. Next, we consider another rational $q'>q$ and apply to it the same procedure applied to q above, to show $q'^2 < 2$. By simple iteration of the procedure, we will get the infinite succession of positive rational numbers q, q', q'', q'''... that belong to A and are all less than 2.

Proof part 2 (for Set B)

Now we may proceed to consider the set $B \subset Q$ that will result in an infinite sequence of rational $q<p$ (Figure 2) which squares q^2 never equals integer 2. The proof follows similar steps as used before for set A where, again, a rational h is introduced.

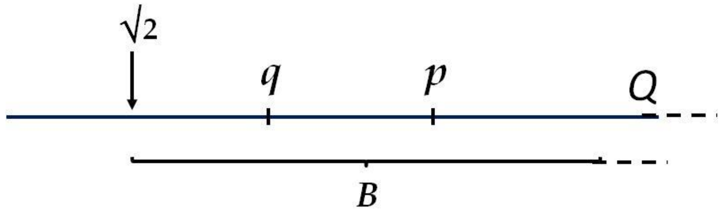

Figure 2

p lies in the set $B \subset Q$ of rational which square p^2 is larger than 2; q is a rational such that $q < p$ and $p - q = h$.

Consider a rational $p \in B$, therefore, p is positive and $p^2 > 2$. Take another rational q such that $q < p$, we need to show that $q^2 > 0$.

Thus, let $q = p - h$, h being a positive rational. We first need to show that $q>0$ to guarantee that the chosen q, although smaller than p, do belong to the cut B (which rational elements must be positive). Because this time $p^2>2$, when writing h we will subtract 2 from p^2 in the numerator, and let h be of the following form:

$$h = \frac{p^2 - 2}{2p}$$

(*c.f.* with the case of set A above, in which $p^2<2$). Thus, since $q=p-h$ we get:

$$q = p - \frac{p^2-2}{2p} = \frac{2p^2-p^2+2}{2p} = \left(\frac{p}{2} + \frac{1}{p}\right) > 0;$$

and being positive the rational q may lie in B. Finally, we may confirm that $q \in B$ by explicitly evaluating q^2:

$$q^2 = \left[p - \frac{p^2-2}{2p}\right]^2 = 2 + \left[\frac{p^2-2}{2p}\right]^2 > 2 \Leftrightarrow q \in B,$$

Thus, we have found a rational q less than p that lies in the set B of the cut. We may apply the mathematical procedure just above to a second rational $q' < q$ to prove that it too belongs to B. By iterating the procedure, we get an infinite descending sequence $q, q', q'',..$ in which all q_i' are in cut B. However, this succession never ends in a rational q which square is $D=2$. Q.E.D.

As stated by Dedekind [2] and considered also by Rudin [1]: "whenever, we have to do with a cut *(A, B)* produced by no rational number, we create a new *irrational* number α, which we regard as completely defined by this cut *(A, B);* we shall say that the number α corresponds to this cut, or that it produces this cut." In the theorem just above the cut corresponds to the irrational $\alpha = \sqrt{2}$ (see Figs. 1, 2)

4. Representing Rational and Irrational Numbers in the Computer

In the computing literature there are various finite precision representations of rational numbers. However, a representation of irrational numbers is generally achieved by approximate solutions based on finite representations, because it is not possible to represent an infinite number of decimal digits with a finite number of 'chunks' of information, be it binary, decimal or any other numerical positive integer radix ß (ß ≥ 2). Because of this, any finite representation of an infinite domain such that of the irrational numbers must have a "rounding function" that maps that domain into a chosen finite set of values that can be represented in the computer [7].

Traditionally, when representing rational numbers *p/q* with $q \neq 0$, both the numerator and denominator are represented in binary using the two's complement convention which allows representation of integer numbers in the range from - 2^{n-1} up to $2^{n-1} - 1$ where *n* is the number of bits in the memory unit. In this type of numerical representation, the most significant bit (or leftmost bit) plays a dual role. First, it indicates the sign of the number, generally 0 for positive and 1 for negative. Second, it participates in computations as any other bit. Due to the presence of zero, the positive numbers that can be represented is one less than the number of negative ones. However, for explanation purposes we will consider only positive integers in the range 0 to $2^{n-1} - 1$ and their binary representation. A typical representation of a *n*-bit memory unit is shown in Figure 3. The numbers in the range 0…(*n*-1) indicate the position of the bits. These numbers are used as the exponents of the binary *base = 2* when the number is expanded to obtain its decimal equivalent.

sign bit	n-2	…	2	1	0

Figure 3

Typical representation of a computer's register in two's complement convention

In this section, we will follow the Kornerup and Matula's notation [7] with a simplification of their approach. Rational numbers are then represented as a two-word encoding. Each word is represented as indicated in Figure 3. Rational arithmetic can be performed as indicated next. In this notation, φ is a rounding function and the rational numbers are irreducible. That is, *gcd(p,q) = 1*

$$\frac{p}{q} \oplus \frac{r}{s} = \varphi\left(\frac{ps+qr}{qs}\right)$$ Addition

$$\frac{p}{q} \ominus \frac{r}{s} = \varphi\left(\frac{ps-qr}{qs}\right)$$ Difference

$$\frac{p}{q} \otimes \left(\frac{r}{s}\right) = \varphi\left(\frac{pr}{qs}\right)$$ Multiplicati

$$\frac{p}{q} \circledast \frac{r}{s} = \varphi\left(\frac{ps}{qr}\right)$$ Division

However, the fixed format of Figure 3 is not the most efficiency for representing very large or small values. A more flexible representation such as the "floating slash" representation has been proposed. Under this new scheme the boundary between the space allocated to these two consecutive words is allowed to "float". The boundary is indicated by a "slash" and hence the name of this representation. The set of representable rational numbers is dependent on both, the radix being used and the number of available digits in a computer word. Under this scheme, at least 2^{k-2} different representations are available in binary. The efficiency of these representations is determined by the ration of irreducible fractions (where gcd(p,q) = 1) to the total of fractions that can be represented. [6, 8].

5. Rational Approximations to Irrational Numbers

As already stated in the introduction it is a well-known fact from elemental arithmetic that every rational number can be expressed as a terminating or periodic fraction [6, 8]. Yet, pure period fractions such as 0.999… o mixed periodic fractions with a period of 9 such as 0.0999…, 0.1999…, 0.2999…, and the like cannot be generated by common fractions (rational) of the form p/q with $q \neq 0$.

However, it is possible to obtain approximations to each of these numbers as we desired. One such procedure, called the *Extended Euclidean Convergent Algorithm* (EECA) provides a sequence of convergent fractions p_i/q_i for i = 0,1…,n where n the value of n is determined by the algorithm itself when a given condition is satisfied. The notion of convergent fractions is based on the definition of continued fractions which can be represented in standard abbreviated notation as $[a_0/a_1/a_2/.../a_n]$ where each $a_i \geq 0$ is an integer number called a partial quotient. As demonstrated in [7, 8] and indicated here without proof "Every rational number can be expressed by a finite simple continued fraction." Partial quotients can be obtained by application of the Euclidean Algorithm. As indicated in [6] and as a justification of the EECA algorithm, an approximation by rational numbers to any irrational number can be justified by Theorem 3 which will be stated here without a formal proof. The interested readers can refer to [6].

Theorem 3 "Given any irrational number λ and any positive integer k, there is a rational number p/q which denominator q does not exceed k such that

$$-1/nk < \lambda - p/q < 1/nk$$

In this work we will use an extension of the Euclidean Algorithm called the Extended Euclidean Convergent Algorithm [7, 8]. This algorithm is shown below in pseudocode and is applied to rational numbers of the form p/q with $q \neq 0$. The algorithm is called Extended because it incorporates the computations of the numerators and denominators of the convergent fractions denoted by p_i and q_i respectively. The assignment operator and the equality operator are indicated following the language R convention of "<-" and "==". In this algorithm, *floor* stands for the function generally known in the mathematical literature as the greatest integer function. The EECA algorithm is presented here in pseudocode which can be easily translated and implemented in any modern programming language.

6. Extended Euclidean Convergent Algorithm (EECA)

Input: integers $p \geq 0$, $q > 0$

Output: sequence integer n and sequences $\{a_i\}$, $\{p_i/q_i\}$ $i = 0,1,\dots n$

Initialization: b_{-1} <- q; b_{-2} <- p; p_{-1} <- 1; p_{-2} <- 0; q_{-1} <- 0; q_{-2} <- 1

i <- 0

repeat

a_i <- floor (b_{i-2}/b_{i-1}) # quotient

b_i <- $b_{i-2} - a_i b_{i-1}$ # remainder

p_i <- $a_i p_{i-1} + p_{i-2}$

q_i <- $a_i q_{i-1} + q_{i-2}$

until $b_i == 0$

Example 1 An application of the EECA algorithm [7]

Let's consider the fraction p/q = 173/55. Table 1 shows the execution of the algorithm. The first two rows are the initial conditions of the algorithm for i = -1 and -2.

i	a_i	b_i	p_i	q_i
-2		173	0	1
-1		55	1	0
0	2	8	3	1
1	6	7	19	6
2	1	1	22	7
3	7	0	173	55

Table 1

In this case, the convergent sequence is obtained by forming the fractions p_i/q_i for i = 0…3. Therefore, the convergent sequence is p_0/q_0 = 3/1 = 3; p_1/q_1 =19/6 ≈ 1.6666…; p_2/q_2 = 22/7 ≈ 3.1423…; p_4/q_4 = 173/55 ≈ 3.1454…. Notice also that the sequence is convergent because each $p_i/q_i < p_{i+1}/q_{i+1}$.

7. Concluding remarks

Whenever the real numbers are introduced via the irrational sets via Dedekind cuts there are always "convenient" numbers which are used without little or no justification. From a pedagogical point of view, these convenient numbers are generally a source of frustration to both students and professors. In this work, the authors have attempted to explain and justify some of the sticky points related to the convenience of these numbers, their nature, and the reason for being so using primarily the infinite ascent method. This method, although a very powerful proving mechanism is, in general, not widely used or at least not mentioned by this name. The authors wanted to illustrate an application of the method using Dedekind cuts. The EECA algorithm is presented to obtain rational approximations to any irrational number via the computer.

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